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The effect of pressure gradient on the law of the wall in turbulent flow

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The effect of a streamwise pressure gradient on the velocity profile in the viscous sublayer of a turbulent flow along a smooth wall in two-dimensional flow is estimated. In the analysis, a similarity argument is used and the necessary empirical information obtained from a constant pressure flow. An allowance is made for the departure from the wall value of the gradient of total shear stress normal to the wall. The results of analysis were used to generate new additive constants for use with Townsend's modified law of the wall velocity profile and subsequently Townsend's profile is found to be in good agreement with the measured velocity profiles in an adverse pressure gradient.

Introduction

It has been known for some time that, close to an impermeable wall, the velocity profiles in a turbulent flow can be non-dimensionalized such that the various profiles measured normal to the wall at different Reynolds numbers collapse on a single curve. This universal curve is termed the law of the wall and has been found to be independent of the streamwise pressure gradient, at least when the flow accelerations involved are not large. This single, well substantiated, empirical law has proven extremely valuable in making predictions of turbulent flow behaviour. For instance, the law of the wall is the corner stone of most turbulent skin friction laws. Coupled to the law of the wake (Coles 1956), the law of the wall provides a means of correlating turbulent boundary-layer velocity profiles which is valid even in moderate streamwise pressure gradients, a fact which has enabled rapid and fairly accurate integral methods of predicting the turbulent boundary-layer behaviour to be developed. In addition, the presence of a universal law of the wall has enabled very simple techniques, such as the Preston tube, to be developed to measure the wall skin friction. Obviously, the law of the wall has proven to be of considerable practical consequence and, as a result, a number of investigators have considered the problem of determining the effect of increasingly large streamwise pressure gradients on this universal velocity profile.

In the fully turbulent flow near the wall, the universal velocity profile given by the law of the wall has a semilogarithmic form, the velocity varying as the logarithm of the distance from the wall. Stratford (1959*a*), however, by using Prandtl's mixing length to relate the mean velocity gradient to the turbulent shear stress and by further assuming a linear distribution of turbulent shear stress near the wall, deduced that when the streamwise pressure gradient was large and the wall skin friction negligible, the velocity would vary as the square root of the distance normal to the wall (the so-called half-power law). The very limited experimental evidence was in quite good agreement with Stratford's predictions. Townsend (1961, 1962) later carefully reappraised Stratford's assumptions and after some modification showed how a closed form velocity profile could be deduced from a mixing length approach. Townsend's profile asymptotes to the logarithmic law of the wall when the pressure gradient (or more precisely the gradient of turbulent shear stress normal to the wall) is small and asymptotes to the half-power law when either the gradient of turbulent stress normal to the wall or the product of this stress gradient and distance from the wall, is large. Three factors have to be determined before Townsend's modified law of the wall can be utilized.

The first, and most easily disposable of the three factors, concerns the coefficient which governs the extent to which turbulent kinetic energy is diffused by pressure and velocity fluctuations. Presumably future experiments will determine this coefficient more precisely, but at the present time it would seem to be somewhat smaller than originally thought by Townsend (see Bradshaw 1965) and consequently, for the present, it can be neglected.

The second factor concerns a constant of integration which arises in Townsend's velocity profile, the constant sometimes being termed either the 'slip velocity' or the 'additive constant'. This additive constant is determined by the flow in the viscous sublayer and, in general, is obtained by matching the velocity at the edge of the sublayer to the fully developed turbulent flow velocity distribution. In his analysis, Townsend assumed that the velocity profile in the viscous sublayer would not be affected by moderate pressure gradients and so suggested an additive constant which is strictly correct only for the zero pressure gradient case.

The third factor to be determined before Townsend's modified law of the wall can be applied concerns the total shear stress gradient normal to the wall; that is, the rate of change with distance normal to the wall of the sum of the laminar and apparent turbulent shear stress. At the wall, the equations of motion show that the gradient of total stress normal to the wall (hereafter termed simply the stress gradient) is equal to the streamwise pressure gradient, and many authors have assumed that this relationship also holds in the fully developed turbulent flow region near the wall. Unfortunately, as Townsend points out, while there is experimental justification for assuming a linear stress distribution in the fully turbulent flow, the stress gradient is, in general, not equal to the pressure gradient and the two gradients can, in many instances, be considerably different. The inequality in the two gradients prescribes, via the equations of motion, the local flow accelerations, and hence the term 'inertia effect' is used to describe the difference between the stress and pressure gradient. Townsend suggests that, under certain circumstances, the stress gradient is determined by conditions pertaining further from the wall. Naturally this added complication is very inconvenient as a great deal of the utility of the law of the wall lies in the fact that it is dependent only on local flow conditions, such as skin friction, and not on the flow history. Townsend's proposal requires that, in order to determine the stress gradient, a running calculation of the flow away from the wall has to be performed, with all the attendant difficulties that this incurs. Indeed in many instances such a running calculation cannot be performed so that the modified law of the wall could not be evaluated.

Recently, in developing numerical solutions to the boundary-layer equations of motion, Mellor (1966*b*) has proposed an eddy viscosity distribution which applies in the viscous sublayer. Outside the sublayer Mellor's suggested distribution corresponds to the conventional distribution of Prandtl's mixing length varying linearly with distance normal to the wall. Mellor further suggested that if the stress gradient were equated to the pressure gradient, then this eddy viscosity distribution could be used to estimate the pressure gradient effect on the additive constant which is missing from Townsend's analysis. However, it is clear that since the stress and pressure gradient are not, in general, equal in the fully turbulent flow adjacent to the sublayer, some doubt must be felt about using Mellor's additive constants, although they might be expected to represent an improvement over the constants assumed by Townsend.

Lastly, Perry, Bell & Joubert (1966) and Perry (1966) have shown that, while both the logarithmic and half-power law of the wall are observed in measured boundary layers in roughly the appropriate areas (the half-power variation of velocity seemingly occurs closer to the wall than might have been expected from Townsend's analysis), Townsend's additive constants apparently do not agree with the measurements. While the findings of Perry *et al.* are open to some criticism, among other things, on the grounds that they have equated the stress and pressure gradients, nevertheless, there did appear to be reason to question the accuracy of Townsend's additive constants. Mellor's (1966*b*) supposedly more accurate additive constants mentioned previously do not satisfactorily explain the discrepancies observed by Perry *et al.* either.

Accordingly, in the present note the whole process of deriving the additive constants was re-examined in an attempt to resolve what was thought to be the observed discrepancies. In the light of the preceding discussion, it was felt that attention ought to be devoted to relaxing the assumption made by Mellor that the stress and pressure gradients are equal in the sublayer, and this has been done.

Analysis

In order to calculate the velocity using a mixing length hypothesis, two quantities must be known. The first is the shear stress and the second the mixing length. When the distributions of these two quantities are known, the velocity profile may be obtained by integration. As a result, the analysis is divided into three sections, the first section being concerned with the mixing length distribution across the sublayer. The second section is concerned with the near-wall stress distribution, and, as mentioned in the introduction, special emphasis is placed on developing a stress distribution which satisfies the wall boundary condition of equal stress and pressure gradient but which is (arbitrarily) linear in the fully turbulent flow outside the sublayer. In the third section, the velocity profile is obtained by numerical integration.

Distribution of mixing length across the viscous sublayer

In the literature, there have been a number of suggested distributions of both mixing length and eddy kinematic viscosity across the viscous sublayer. These suggestions have been reviewed by both Rotta (1962) and Hinze (1959) and objections raised on one count or another against most of the proposals. More serious, perhaps, from the present point of view, is that almost all of the suggested distributions suppose a unique relationship between the mixing length or eddy viscosity and the law of the wall similarity co-ordinate normal to the wall. This leads to obvious difficulties in the case of Stratford's (1959*a*) zero wall stress boundary layer, a boundary layer which in the present note is regarded as the asymptotic case of departure from the law of the wall. In Stratford's case, the law of the wall co-ordinate is zero throughout the layer. Evidently, a mixing length or eddy viscosity distribution based on a law of the wall co-ordinate must be rejected in the present circumstances if the proper asymptotic behaviour of the modified law of the wall is to be obtained.

In his approach, Mellor (1966*b*) suggested a new similarity co-ordinate for the eddy viscosity distribution normal to the wall. This new similarity co-ordinate reduces to the law of the wall co-ordinate when the pressure gradient is small and does not degenerate to zero when the wall friction disappears. The eddy viscosity distribution across the sublayer is subsequently derived from velocity profile measurements in a constant pressure flow and expressed only as a function of the new similarity co-ordinate. The resulting distribution of eddy viscosity is supposed valid when the streamwise pressure gradient is nonzero. In deriving this distribution, Mellor used only dimensional arguments, pointing out that alternative similarity co-ordinates do not correspond to the velocity profile similarity behaviour observed in the absence of a pressure gradient. While the correct behaviour in the absence of a pressure gradient obviously fulfills a necessary condition of any new similarity co-ordinate, unfortunately it is not a sufficient condition. It is unlikely that, in the present state of experimental and theoretical knowledge of the turbulent flow near a wall in the presence of large pressure gradients, the question of the proper similarity co-ordinate can be rigorously resolved. However, it is found in the present analysis that a slightly different argument, based on the mixing length concept, leads to the same similarity co-ordinate but with perhaps a clearer indication of the approximations involved. As in Mellor's approach, the zero pressure gradient experimental evidence is used to derive the variation of the von Kármán constant, κ (and hence the mixing length distribution) which, when expressed in terms of the new similarity co-ordinate, is supposed independent of pressure gradient. In practical terms, the differences between velocity profiles obtained using either mixing length or eddy viscosity distributions are negligible provided the stress distributions are the same in both cases, since both Mellor's and the present distribution are derived from, and adjusted to reproduce, the same zero pressure gradient data. The present author simply prefers to work in terms of a mixing length since it would seem to have a better physical basis, especially when interpreted as the dissipation length scale of the turbulent kinetic energy (Townsend 1961).

The velocity distribution near the wall is functionally expressed as

$$u = f(y, \nu, \rho, \tau, \partial\tau/dy, \ldots), \tag{1}$$

where τ, the total shear stress is defined as

$$\tau = \rho\left[\nu\frac{\partial u}{\partial y} - \overline{u'v'}\right] \tag{2}$$

with ρ and ν, the fluid density and kinematic viscosity, supposed constant throughout the flow. The velocities in the x and y directions, that is parallel to and normal to the wall, are denoted by u and v respectively. A prime denotes a fluctuation and a bar a time mean average. It is supposed that only the flow variables given in (1) influence the velocity field and the justification for this assumption must reside in the observed velocity profile behaviour. This point will subsequently be briefly considered further but for the moment (1) is accepted. If the stress gradient $\partial\tau/\partial y$ is near constant ($\simeq \tau/y$), or at least expressible in terms of y, ν, ρ and τ, then

$$u = u_s f\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}}\right), \tag{3}$$

where u_s is some scale of velocity. The conventional law of the wall similarity co-ordinate is obtained from (3) by assuming that the total stress is constant across the wall region and this gives

$$u^+ = f(y^+), \tag{4}$$

where

$$\left.\begin{aligned} y^+ &= yu_\tau/\nu, \\ u^+ &= u/u_\tau, \\ u_s = u_\tau &= (\tau_w/\rho)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5}$$

When the total stress is not constant across the wall region, Mellor's similarity co-ordinate may be obtained from (1) again by either assuming a constant stress gradient or a stress gradient which is a function only of y, ν, ρ and τ, and in addition noting that τ may be replaced in (1) by $\partial u/\partial y$. With these assumptions (1) becomes

$$u = f(y, \nu, \rho, \partial u/\partial y), \tag{6}$$

which gives

$$u = u_s f\left\{y\left(\frac{1}{\nu}\frac{\partial u}{\partial y}\right)^{\frac{1}{2}}\right\}. \tag{7}$$

Thus a new similarity co-ordinate, Y, may be defined as

$$Y = y\left(\frac{1}{\nu}\frac{\partial u}{\partial y}\right)^{\frac{1}{2}}, \tag{8}$$

where $(\kappa_0 Y)^2$ is identical to the similarity parameter ζ suggested by Mellor (1966*b*). The velocity scale u_s is at present undefined. It is evident from (2) that replacement of τ by $\partial u/\partial y$ in (1) is valid at least very close to the wall where $\overline{u'v'}$ is negligible (the wall boundary conditions also suggest a local constant stress gradient in this same region). In addition at the other extreme, in the fully turbulent wall flow where the viscous contribution to the total stress may be neglected, it is thought that both a mixing length concept and a constant stress gradient are reasonable assumptions (Stratford 1959*a*). Since the mixing length concept gives

$$-\overline{u'v'} = \left(l\frac{\partial u}{\partial y}\right)^2 = \left(\kappa y\frac{\partial u}{\partial y}\right)^2, \tag{9}$$

hence when κ, the von Kármán constant, is independent of y, as indeed it appears to be in the fully turbulent near wall flow, it is a reasonable assumption to replace τ by $\partial u/\partial y$ in (1). At any rate, later in the present note the turbulent velocity profiles predicted by the linear mixing length and stress distribution assumptions will be evaluated so that this point may be checked. However, it would appear, on the basis of the presently available information, that (7) would be valid both at the inner edge of the viscous sublayer and in the fully turbulent flow and consequently the choice of Y for the similarity co-ordinate of the entire wall region is not unreasonable.

Before proceeding, it is interesting to note that if the constant stress gradient were equated to the pressure gradient then (1) would be identical to the similarity proposal of Perry *et al.* (1966). Perry *et al.* subsequently verified their similarity proposal using experimental evidence and therefore this verification adds weight to the present work. However, two restrictions on this experimental verification must be mentioned. The first is that the experimental evidence is generally available only for either weak pressure gradients or for the fully developed turbulent flow above the viscous sublayer. The second restriction is that more recently Perry (1966) has interpreted certain experimental evidence as suggesting an effect of the streamwise rate of change of pressure gradient on the velocity profile. In the present analysis this amounts to suggesting the inclusion of a $\partial^2\tau/\partial x\,\partial y$ term in (1). Perry's interpretation is disputed later in the present note on the basis of additional experimental evidence. Nevertheless, the possibility that, when the mean flow in the sublayer is changing rapidly in the streamwise direction, additional terms ought to be included in (1) cannot be ruled out. As has been pointed out by one of the referees, $f(Y)$, without loss of generality, may be related to $f\{(-\overline{u'v'})^{\frac{1}{2}}y/\nu\}$, that is to a y^+ based on the local Reynolds stress. Written in this form it is not difficult to doubt that the turbulence in the sublayer is really specified by only a local value of this turbulence Reynolds number.

To continue, if the similarity co-ordinate Y is accepted then the von Kármán constant κ, which must vary across the sublayer, may be expressed as a function of Y only. The appropriate form of this function is obtained from the zero pressure gradient sublayer velocity profile. Using (2), (5), (8) and (9) the velocity profile gradient is written

$$\frac{\partial u}{\partial y} = \frac{\tau/\rho\nu}{1+\kappa^2 Y^2}. \tag{10}$$

Equation (10) can be solved for κ in the zero pressure gradient case since in this instance the total stress is nearly constant across the layer and the mean velocity profile known quite accurately from measurements. Hence (10) yields the empirical result that

$$(\kappa/\kappa_0)^2 \begin{cases} = 0{\cdot}00714Y + \exp\{3(Y-9)\} & (Y < 9), \\ = 1 & (Y \geqslant 9), \end{cases} \tag{11}$$

where κ_0 is the conventional von Kármán constant, appropriate to the fully developed turbulent flow. Equation (11) is plotted in figure 1 and has been arranged to give κ varying with $y^{\frac{1}{2}}$ in the region very close to the wall in keeping

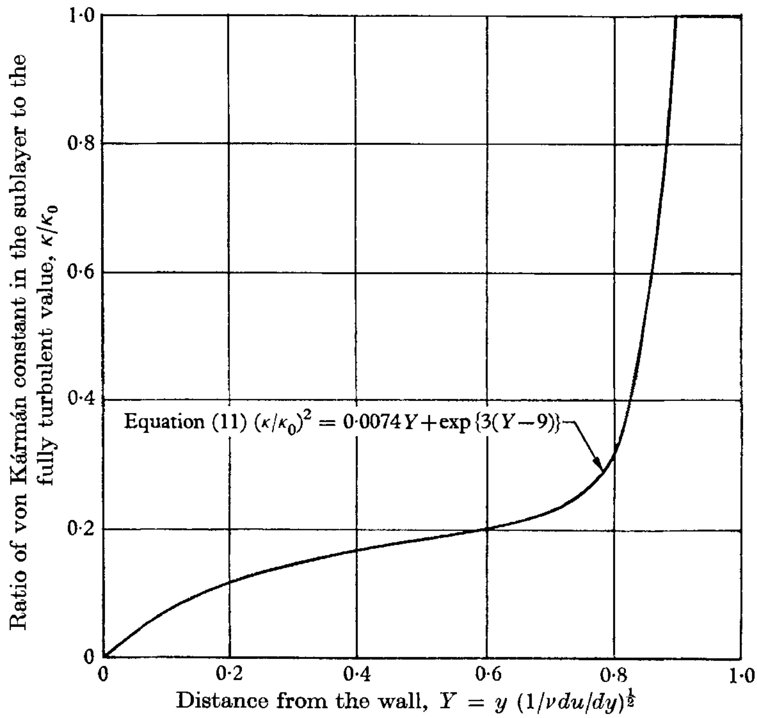

FIGURE 1. Distribution of the von Kármán constant across the sublayer.

with Townsend's (1956) remarks that $\overline{u'v'}$ should vary with y^3 in the region of the wall. The problem of determining the distribution of total shear stress in the viscous sublayer is now examined.

Distribution of shear stress across the viscous sublayer

The first boundary condition imposed on the total shear stress is obtained from the equations of motion together with the wall no-slip condition, and requires that the stress gradient normal to the wall and the streamwise pressure gradients be equal at the wall. However, on the basis of experimental evidence (see for instance Sandborn & Slogar 1955; Spangenberg, Rowland & Mease 1967; Bradshaw 1966) it is known that the stress and pressure gradients quickly become unequal as distance from the wall is increased.

Mellor (1966*b*), however, did demonstrate theoretically that, if this equality of stress and pressure gradient were assumed, considerable departures from the constant pressure velocity profiles could be obtained in the presence of large streamwise pressure gradients, even across the viscous sublayer. On the basis of Mellor's findings, it would seem that the additive constant in Townsend's modified law of the wall, obtained by matching the velocity at the sublayer edge to the modified law of the wall, would depend on pressure gradient when the pressure gradients were large. In view of the utility of the modified law of the wall, it is therefore of considerable interest to see if Mellor's findings are substantiated when a more realistic stress distribution is specified.

On the basis of Schubauer & Klebanoff's (1951) measured turbulent stress distributions, Townsend (1961, 1962) was led to suggest that near the wall the stress gradient was indeed constant, but not equal to the streamwise pressure gradient. The additional experimental evidence of Newman (1951), Sandborn & Slogar (1955), Bradshaw (1966) and Spangenberg *et al.* (1967) all support Townsend's suggestion of a constant stress gradient in the wall region. Thus it would appear legitimate to require the stress distribution to become linear as distance from the wall is increased but not to require that the gradient of this linear distribution be equal to the streamwise pressure gradient.

The simplest stress distribution which can be suggested is a Taylor series expansion in terms of the wall boundary conditions. Unfortunately such a series is not convergent at sufficiently large distances from the wall for the present purposes. An alternative series having the required properties is given by

$$\frac{\partial \tau}{\partial y} = \sum_{n=0}^{n} a_n \tanh^n \eta y, \tag{12}$$

where the a_n may be determined from the wall boundary conditions as

$$a_0 = (\partial \tau/\partial y)_w = dp/dx, \tag{13}$$

$$a_1 = (\partial^2 \tau/\partial y^2)_w = 0, \tag{14}$$

$$a_2 = (\partial^3 \tau/\partial y^3)_w = (\tau_w d\tau_w/dx)/(2\rho\nu^2\eta_w^2), \tag{15}$$

and so on. From dimensional considerations the argument of the hyperbolic tangent in (12) must be dimensionless and in addition, if Y is supposed to be the wall layer similarity co-ordinate, then

$$\eta y = f_1(Y). \tag{16}$$

It is convenient to introduce the parameter b defined by

$$\eta y = bY, \tag{17}$$

so that using (8) and (10) gives

$$b/b_w = \eta/\eta_w(1+\kappa^2 y^2)^{\frac{1}{2}} \tag{18}$$

and
$$\eta_w = b_w(\tau_w/\rho\nu^2)^{\frac{1}{2}}, \tag{19}$$

where the subscript w indicates a value when y is zero. Thus the search is for the distribution of b as a function of Y so that (12) can be used to provide the stress gradient. As before, the precise form of the functional dependence of b on Y is empirically obtained from the measured velocity profiles in the sublayer as follows: Coles (1956), among others, has shown that a law of the wall in the form of $u^+ = f(y^+)$ when inserted into the equations of motion gives the stress gradient as

$$\partial\tau/\partial y = dp/dx + \rho u_\tau u'_\tau u^{+2}, \tag{20}$$

where du_τ/dx has been written as u'_τ. This stress gradient may be compared with the stress gradient obtained from (12) and since it is known empirically that the sublayer velocity profile in the absence of pressure gradients is given quite closely by

$$u^+ = 14 \tanh(y^+/14), \tag{21}$$

(suggested by Rannie (1956) and demonstrated by Hinze (1959)) it follows that empirically $\eta = \eta_w$ (= a constant) and in the case where the velocity profile is given by (21) that is, when $dp/dx \approx 0$, b_w has a value of 1/14. Hence placing $\eta = \eta_w$ in (18) permits the stress gradient to be specified in terms of Y, the wall boundary conditions and the presently undetermined constant b_w.

In the light of Townsend's (1961, 1962) work, it would be expected that the (constant) stress gradient occurring in the fully turbulent flow would, in many instances, be the result of conditions pertaining further from the wall. Necessarily then, the asymptotic constant gradient given by (12) for $bY > 2{\cdot}0$, say ,would have to equal this specified gradient, and this allows b_w to be determined. Viewed in this manner, the constant b_w serves to fair any given linear stress distribution into the wall shear stress, τ_w, such that the wall boundary conditions are satisfied in a reasonable manner. Conversely, when the pressure gradient is small and historic effects on the stress distribution negligible, the constant pressure value of b_w could serve to relate the stress and pressure gradient.

The calculated velocity profiles in the region of the wall

Some computational and non-dimensional considerations. Since the mixing length and total stress distribution in the wall region are now considered known, the velocity profile can be obtained by integration of (10). The details of this procedure are now given.

From (12), the total stress gradient is

$$\frac{\partial\tau}{\partial y} = a_0 + a_1 \tanh By + a_2 \tanh^2 bY + \dots. \tag{22}$$

For values of the argument bY greater than about 2·0 the hyperbolic tangent has a value of almost unity, and so in this region the stress gradient is nearly constant. From (22), this gives

$$\begin{aligned}\frac{\partial\tau}{\partial y} &= a = a_0 + a_1 + a_3 + \dots \\ &= \frac{dp}{dx} + \frac{1}{2b_w^2}\frac{d\tau_w}{dx} + \frac{1}{3b_w^3}\frac{\nu}{u_\tau}\frac{d^2p}{dx^2} + \dots\end{aligned} \tag{23}$$

which integrates to give

$$\tau = ay + \tau_0, \tag{24}$$

where τ_0 is the intercept of the linear total shear stress distribution with the $y = 0$ axis. Since the stress gradient in the sublayer is not equal to a, τ_0 is not equal to τ_w, the wall shear stress, although τ_0 is not expected to be significantly different from τ_w, except when the stress and press gradients are considerably different and the pressure gradient is large. In his analysis, Townsend (1961, 1962) has equated τ_0 to τ_w. Continuing, for $bY > 2$, the velocity gradient is given by (24) and (10), on neglecting the laminar stress contribution in the fully turbulent régime, as

$$\frac{\partial u}{\partial y} = \frac{(ay+\tau_0)^{\frac{1}{2}}}{\rho^{\frac{1}{2}}\kappa_0 y}, \tag{25}$$

which integrates to

$$\left.\begin{aligned} u &= \frac{2}{\rho^{\frac{1}{2}}\kappa_0}(ay+\tau_0)^{\frac{1}{2}} + \left(\frac{\tau_0}{\rho\kappa_0^2}\right)^{\frac{1}{2}} \ln\left|\frac{(ay+\tau_0)^{\frac{1}{2}}-\tau_0^{\frac{1}{2}}}{(ay+\tau_0)^{\frac{1}{2}}+\tau_0^{\frac{1}{2}}}\right| + B \quad (\tau_0 > 0, \quad ay+\tau_0 > 0) \\ &= \frac{2}{\rho^{\frac{1}{2}}\kappa_0}(ay+\tau_0)^{\frac{1}{2}} - 2\left(\frac{-\tau_0}{\rho\kappa_0^2}\right)^{\frac{1}{2}} \tan^{-1}\left\{\frac{(ay+\tau_0)^{\frac{1}{2}}}{(-\tau_0)^{\frac{1}{2}}}\right\} + B \quad (\tau_0 < 0, \quad ay+\tau_0 > 0), \end{aligned}\right\} \tag{26}$$

where
$$B = u_j - \frac{2}{\rho^{\frac{1}{2}}\kappa_0}(ay_j+\tau_0)^{\frac{1}{2}} - \left(\frac{\tau_0}{\rho\kappa_0^2}\right)^{\frac{1}{2}} \ln\left|\frac{(ay_j+\tau_0)^{\frac{1}{2}}-\tau_0^{\frac{1}{2}}}{(ay_j+\tau_0)^{\frac{1}{2}}+\tau_0^{\frac{1}{2}}}\right| \tag{27}$$

and so on in the case $\tau_0 > 0$. The subscript j indicates some point where $bY > 2$, and B is the aforementioned additive constant. Equation (26) is, of course, Townsend's (1961) modified law with the slight difference that τ_0 has not been equated to τ_w, the wall stress.

In the sublayer, the velocity profile is obtained by integration of (10), using (8), (11), (18) and (22), given values of the boundary conditions and b_w. Simple analytic integration of these expressions was not found possible, so a numerical procedure was adopted and programmed for the Univac 1108. Integration was carried out in the Y-plane out to a Y of 10. At this station, the now linear stress distribution was extrapolated back to $y = 0$ to give τ_0. Using this value of τ_0, equation (26) was evaluated at $Y = 10$ to give B.

Two non-dimensional velocity profile presentations are pertinent to the present study. The first is, of course, the conventional law of the wall presentation of u^+ *vs.* y^+ and the second corresponds to Mellor's (1966*b*) pressure velocity u_p which, as in Mellor's note, is used to define a u^* and y^* as follows

$$u^* = u/u_p, \quad y^* = yu_p/\nu, \tag{28}$$

where
$$u_p = (\nu a_0/\rho)^{\frac{1}{3}} \tag{29}$$

and u_p, the pressure velocity is analogous to u_τ, the frictional velocity. The non-dimensional form of the modified law of the wall in conventional co-ordinates becomes

$$\begin{aligned} u^+ = \frac{A^{\frac{1}{2}}}{\kappa_0}\ln\left|\frac{4A}{\alpha}\left\{\frac{(\alpha y^+ + A)^{\frac{1}{2}} - A^{\frac{1}{2}}}{(\alpha y^+ + A)^{\frac{1}{2}} + A^{\frac{1}{2}}}\right\}\right| \\ + \frac{2}{\kappa_0}[(\alpha y^+ + A)^{\frac{1}{2}} - A^{\frac{1}{2}}] + B^+ \quad (\alpha y^+ A > 0, \quad A > 0), \end{aligned} \tag{30}$$

where
$$\left.\begin{aligned} \alpha &= a\nu/\rho u_\tau^3 \\ &= \alpha_0 + \alpha_2 + \alpha_3 + \dots, \\ A &= \tau_0/\tau_w, \\ B^+ &= B/u_\tau - \frac{A^{\frac{1}{2}}}{\kappa_0}\ln\left|\frac{4A}{\alpha}\right| + \frac{2A^{\frac{1}{2}}}{\kappa_0}. \end{aligned}\right\} \tag{31}$$

Obviously, (30) is not a suitable form of the velocity profile when the wall shear stress is small (or the pressure gradient very large), so in this case the velocity profile is expressed as

$$u^* = \frac{2}{\kappa_0}[(Cy^*+A^*)^{\frac{1}{2}} - A^{*\frac{1}{2}}] + \frac{A^{*\frac{1}{2}}}{\kappa_0}\ln\left|\frac{4A}{\alpha}\left\{\frac{(Cy^*+A^*)^{\frac{1}{2}} - A^{*\frac{1}{2}}}{(Cy^*+A^*)^{\frac{1}{2}} + A^{*\frac{1}{2}}}\right\}\right| + B^*, \quad (32)$$

where

$$\left.\begin{aligned} B^* &= B/u_p + \frac{2}{\kappa_0}A^{*\frac{1}{2}} - \frac{A^{*\frac{1}{2}}}{\kappa_0}\ln\frac{4A}{\alpha} = \frac{B^+}{\alpha_0^{\frac{1}{3}}}, \\ C &= a/a_0 = \alpha/\alpha_0, \\ \alpha_0 &= (u_p/u_\tau)^3 = (\nu a_0/\rho u_\tau^3), \\ A^* &= A\alpha_0^{-\frac{2}{3}}. \end{aligned}\right\} \quad (33)$$

Similar non-dimensional forms can be obtained from (26) when $\tau_0 < 0$. Placing $A = 1{\cdot}0$ in (30) results in the law of the wall in the form given by Townsend (1961), while placing $A = 1, C = 1$ in (32) gives the law of the wall in the second of the two forms given by Mellor (1966*b*). The results of the calculations are now presented in these non-dimensional forms and discussed.

The calculated velocity profiles. Velocity profiles were computed for a wide range of pressure gradients and are shown in figures 2 and 3 for the case where the stress gradient α and pressure gradient α_0 are equal, the case treated by Mellor (1966*b*). In performing the calculations the asymptotic value of the von Kármán constant κ_0 was taken to be 0·41. Not unexpectedly, these computed profiles were found to be in excellent agreement with Mellor's calculations in all instances. In figure 4 velocity profiles are shown for three typical streamwise pressure gradients α_0, where for each of the three pressure gradients, three stress gradients α have been assumed. The computed profiles show clearly that the variation in total stress gradient across the sublayer has only a very small effect on the sublayer velocity profile. The principal reason for departure from the law of the wall in the sublayer is thus seen to result from the direct effect of the pressure gradient. One proviso must be added to the previous statement and this concerns the parameter b_w, which will now be discussed.

The constant b_w enters the calculation of the velocity profile from two different ways. First, the relationship between the stress and pressure gradient serves to determine b_w. Obviously this condition itself does not influence the previous finding concerning the sublayer velocity profile. However, the numerical value of b_w enters the calculation in a more subtle manner, for it may be recalled that the parameter b_w governed the distribution of shear stress across the sublayer and consequently to specify b it was necessary to introduce a value for b_w. In the calculations presented in figure 4 the constant pressure value of b_w of 1/14 was used. Subsequently it was found that when the pressure gradient was large, a change in b_w of 11·5 % produced a change in calculated u^+ of less than 0·5 % at a y^+ of 150. This small change in u^+ is obviously negligible so it may be assumed that the results presented in figure 4 are insensitive to the actual numerical value taken for b_w and thus have a general validity.

The finding that the so-called inertia effects are small in the sublayer does not imply that the additive constant B^+ and the stress ratio constant A depend only

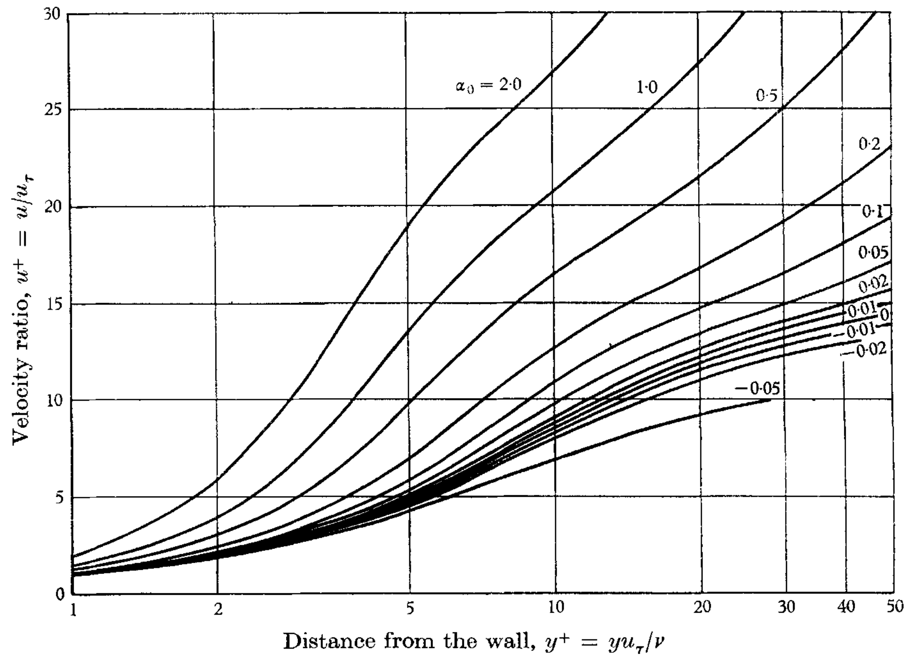

FIGURE 2. Effect of pressure gradient on the velocity profile near the wall. A semi-logarithmic plot. Stress gradient α = pressure gradient $\alpha_0 = (\nu/\rho u_\tau^3)/(dp/dx)$.

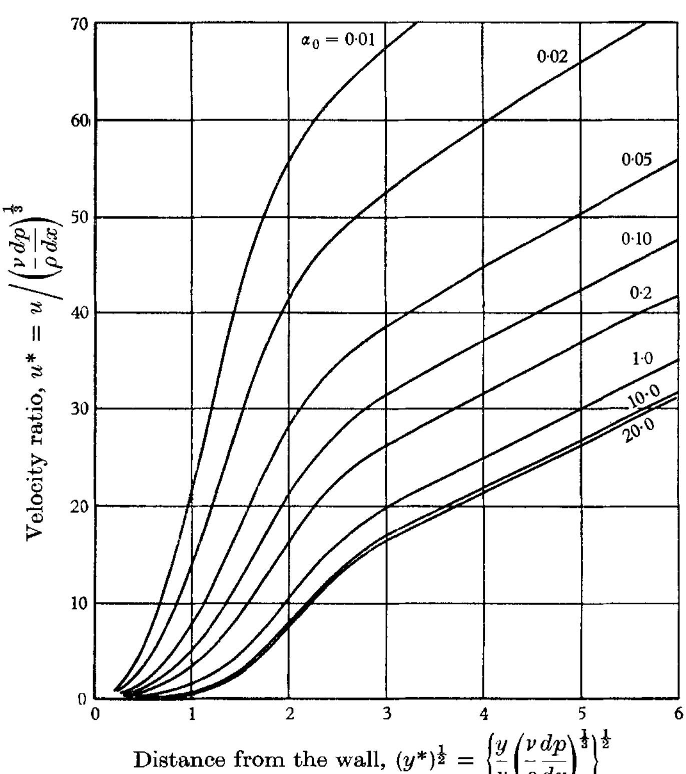

FIGURE 3. Effect of pressure gradient on the velocity profile near the wall. A square root plot. Stress gradient α = pressure gradient $\alpha_0 = (\nu/\rho u_\tau^3)(dp/dx)$.

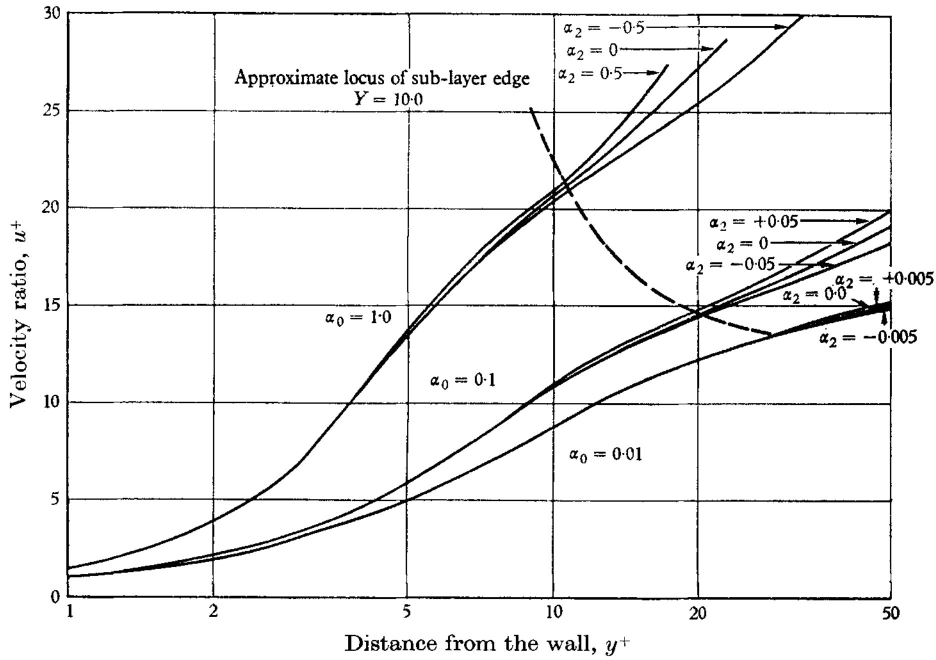

FIGURE 4. Effect on the velocity profile of the variation in the stress gradient across the sublayer.

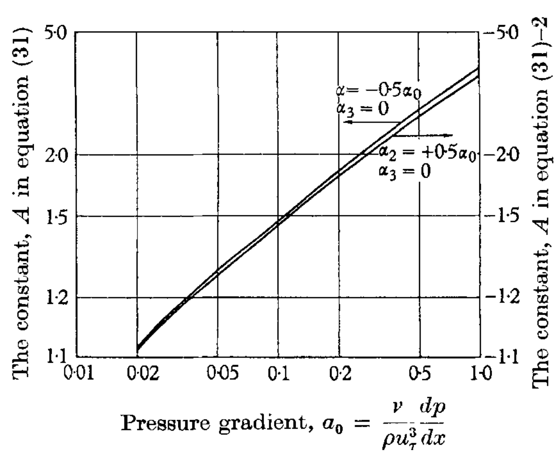

FIGURE 5. Effect of stress gradient on the constant A. Note that when $\alpha_2 = 0, A = 1{\cdot}0$ and also when $\alpha_0 = 0$, $A = 1{\cdot}0$.

on the pressure gradient. Indeed the opposite is true; that is, since the velocity at the edge of the sublayer does not vary appreciably with the stress gradient α when the pressure gradient α_0 is fixed, it follows from their definitions that A and B^+ must vary with the stress gradient. A and B^+ are readily obtained from the

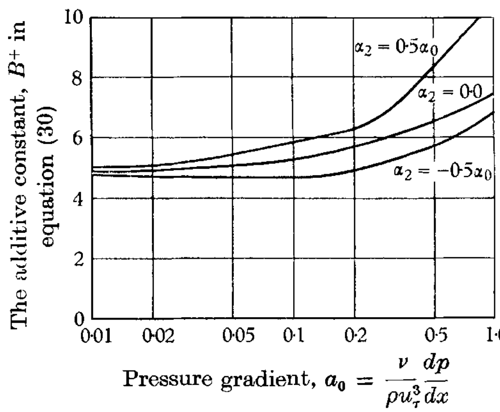

FIGURE 6. Effect of pressure gradient on the additive constant B^+.

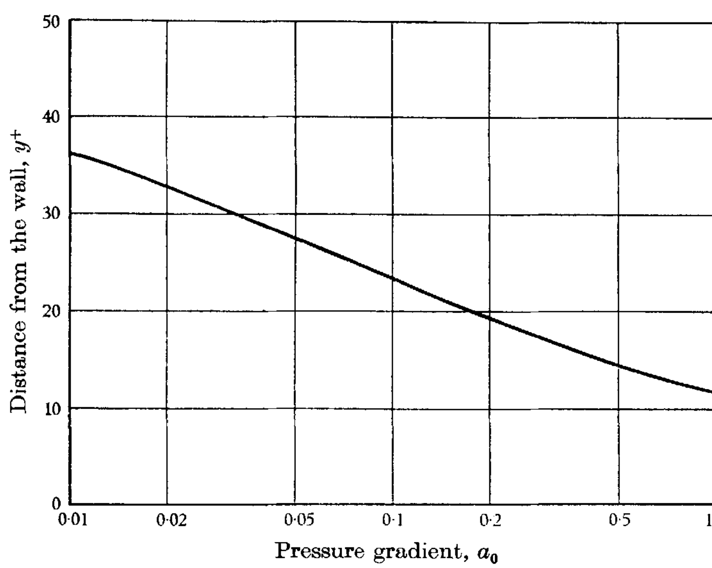

FIGURE 7. Effect of pressure gradient on the location of the sublayer edge ($Y = 10{\cdot}0$). Stress gradient α = pressure gradient α_0. $\alpha_2 = \alpha_3 = 0$, $Y = y^+(\partial u^+|\partial y^+)^{\frac{1}{2}} = 10$.

computed velocity profiles and their variation with stress and pressure gradient is presented in figures 5 and 6. Obviously approximate values of A and B^+ may be obtained from any standard set of constants by assuming that the velocity at the edge of the sublayer u^+, say at a Y of $10{\cdot}0$, depends only on pressure gradient. To enable this procedure to be carried out it is necessary to have an estimate of the location of the outer edge of the sublayer in y^+ co-ordinates and this is presented in figure 7.

Having established sublayer velocity profiles the next step is to use these profiles in conjunction with the modified law of the wall and to note any discrepancy between prediction and experiment.

Comparisons with measured profiles

A great difficulty arises in attempting to compare predicted and measured turbulent boundary-layer velocity profiles near a wall. This difficulty arises because the wall shear stress τ_w must be known before any meaningful comparisons can be made. In the literature, there is apparently not one case where significant pressure gradient effects on the law of the wall could be expected and where both the velocity profile and skin friction have been measured directly. Indirect measurements of skin friction have been made, using such devices as the Preston or Stanton tube or a sublayer fence, in circumstances where a departure from the law of the wall might be expected. In the most reliable cases, the Preston or Stanton tube used was checked against a sublayer fence (see Patel 1965), on the assumption that the sublayer fence calibration would not be subject to a pressure gradient effect, since, so it was thought, the velocity profile in the sublayer would be insensitive to the pressure gradient. The result of both the present and Mellor's (1966*b*) investigation would indicate that the velocity profile in the sublayer could vary from the constant pressure distribution by a significant amount when the pressure gradient was large. It would therefore seem that the sublayer fence with a zero pressure gradient calibration could not be used to measure skin friction in the presence of large pressure gradients. For instance, on the basis of figure 2, when the pressure gradient parameter, α_0, is 0·1 (a large but not improbable pressure gradient), a zero pressure gradient calibration would indicate a friction velocity u_τ approximately 1·25 times too large if the effective centre of the device were at y^+ of 10. This, in turn, would lead to an indicated skin friction of more than 1·5 times the actual friction if the present work is to be believed. In addition, the pressure gradient parameter, α_0, is dependent on the cube of the friction velocity, so if the friction velocity were in error by 25 % it would be then thought that the pressure gradient parameter, α_0, was 0·05 instead of 0·1. The only fortunate feature of these findings is that when eventually the sublayer fence (or any similar skin friction meter operating submerged in the sublayer) is calibrated for use in a large pressure gradient, the additional calibration factor will be the pressure gradient parameter, α_0, and not, as could so easily have been the case, the stress gradient parameter, α. However, the problem remains as to what to do concerning the available velocity profiles and how to compare them with predictions.

If a rigorous course of action were to be pursued, the comparison between theory and experiment would have to be abandoned at this point. Approximate corrections to the skin friction to account for the pressure gradient can, of course, always be devised, but to a considerable degree these corrections beg the question of the effect of pressure gradient on the law of the wall. However, for the small pressure gradient case, $\alpha_0 > 0{\cdot}02$, the velocity profile close to the wall in the region of $y^+ = 50$ is not appreciably different from the conventional law of the wall, as may be seen from figure 2. Accordingly, skin friction coefficients determined by one way or another, assuming the conventional constant pressure law to hold in this region, would not be expected to be significantly in error. At the other extreme, when the pressure gradient parameter, α_0, is very large, $\alpha_0 > 1{\cdot}0$, the theoretical profiles are insensitive to the skin friction, as can be seen from figure 3; skin friction in this co-ordinate frame only influences α, and not the co-ordinates themselves. As a result, in all but one test series, the values of skin friction quoted by the actual investigator were taken at face value. The one exception was in the case of Newman's (1951) measurements where, as the boundary-layer developed towards separation, sufficient, apparently quite accurate, hot-wire measurements of turbulent shear stress were available so that an extrapolation down to the wall could be performed. Newman did, in fact, perform this extrapolation but in doing so used the condition that close to the wall the stress and pressure gradients would be equal. As has been pointed out frequently in the present note, this condition only holds at the wall and not close to the wall, particularly in these cases where the pressure gradient is changing rapidly. The present author re-performed the extrapolation at stations D, E, F and G without neglecting the inertia effects, and obtained slightly different answers for the wall stress. These new values for skin friction lay approximately mid-way between Newman's extrapolated values and the values obtained from the Ludwieg–Tillman skin friction law.

The problem of determining the stress gradient, α, pertaining to a measured velocity profile, further complicates matters. In most cases this stress gradient is simply not known. Where possible an estimate of the stress gradient was made in accordance with (23) by graphically differentiating the skin friction distributions and using a value of b_w of 1/14. In cases where the skin friction distribution was not known, the best that could be done was to present theoretical profiles for the possible extreme conditions that the stress gradient could have had. In view of all these unknowns, there did not seem to be much point in making prolonged comparisons between theoretical and measured profiles, and only a fairly limited sample is presented.

First of all, an almost random selection of six of the more recently measured velocity profiles in an adverse pressure gradient was made. These profiles were obtained from the work of Perry (1966), Spangenberg *et al.* (1967), Patel (1965) and Stratford (1959*b*) and were all plotted in terms of u^* and y^*, since these co-ordinates do not involve a knowledge of the surface shear stress. In addition, the main area of interest is not in the region very close to the wall where the velocity distribution is sometimes not even appreciably different from the law of the wall, but in the region further from the wall where significant departures from the law of the wall have had the opportunity to develop. Since these departures from the logarithmic form have a square root dependence, it seems only natural to plot the profiles on a square root abscissa.

In the experiment of Perry (1966), no measurements of either surface shear or the turbulent shear stress distribution normal to the wall were made. An estimate of the surface shear was obtained by Perry by fitting the measured profiles to the logarithmic law of the wall, and the present author obtained an estimate of the stress gradient by using $b_w = 1/14$ in (23), together with a graphical evaluation of the experimental $d\tau_w/dx$. The resulting comparison between Perry's measurements and the theory is shown in figure 8. As mentioned earlier in deriving this modified law of the wall, Townsend (1961) included the effect of pressure velocity

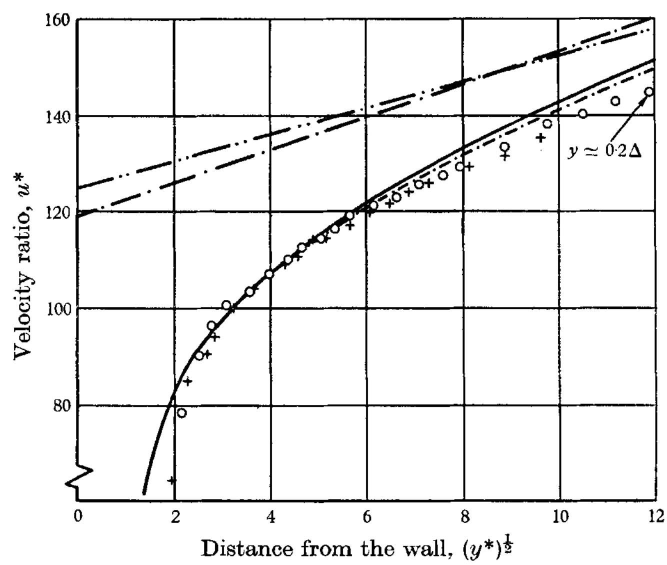

FIGURE 8. A comparison between measured velocity profiles of Perry and theory. Theory: $\alpha_0 = 0{\cdot}0037$; ———, $\alpha = 0{\cdot}00256$, $\bar{B} = 0$; —·—·—, $\alpha = 0{\cdot}00256$, $\bar{B} = 0{\cdot}2$. Asymptotic half power law: —·—, $\alpha = 0{\cdot}00256$, $\bar{B} = 0$; —··—, $\alpha = 0{\cdot}00256$, $\bar{B} = 0{\cdot}2$. Experiments of Perry (1966); $\alpha_0 = 0{\cdot}0037$; ○, station $x = 2{\cdot}5$ ft.; +, station $x = 4{\cdot}0$ ft.

diffusion of turbulent kinetic energy, and the effect of this additional term is also shown in figure 8. In calculating the pressure-velocity diffusion contribution, Townsend's suggested value for the governing coefficient $\bar{B}$ (Townsend's 'B') of 0·2 was taken. In all the calculations performed, the sublayer was supposed unaffected by this additional pressure-velocity diffusion contribution. It will also be recalled that the asymptotic form of Townsend's modified law of the wall, (26), when the product ay is much larger than τ_0 is simply

$$u = \frac{2}{\rho^{\frac{1}{2}}\kappa_0}(ay)^{\frac{1}{2}} + B. \tag{34}$$

These asymptotic profiles have been evaluated for both cases of negligible and of non-negligible pressure velocity diffusion of turbulent kinetic energy and are also shown in figure 8.

In the experiments of Spangenberg *et al.* (1967), the turbulent shear stress gradient could be deduced from direct measurements of the shear made using a hot-wire anemometer. The surface shear stress was in this case, however, obtained by fitting the measured velocity profiles to the law of the wall. Nevertheless, taking the measured gradients and the quoted wall stress, the resulting comparisons between theory and experiment are shown in figure 9. Also shown in figure 9 is the effect of allowing for pressure-velocity diffusion of turbulent kinetic energy, together with an illustration of the effect of equating the stress and pressure gradients. An asymptotic velocity profile is shown, for comparative purposes, in the case of equal stress and pressure gradients.

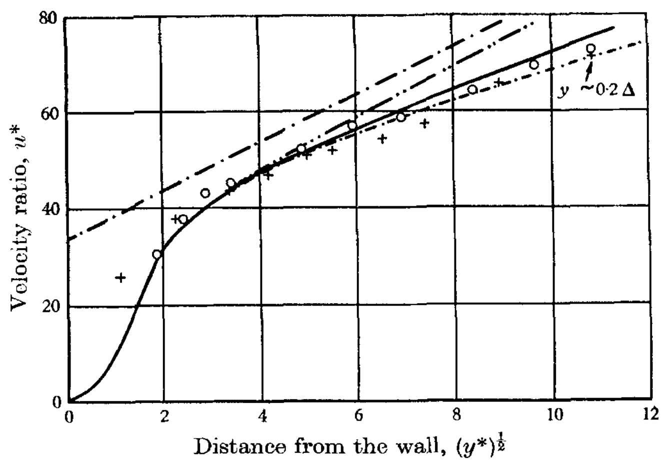

FIGURE 9. A comparison between measured velocity profiles of Spangenberg *et al.* and theory. Theory: $\alpha_0 = 0{\cdot}04$; —··—, $\alpha = 0{\cdot}04$; ———, $\alpha = 0{\cdot}02$, $\bar{B} = 0{\cdot}0$; —·—·—, $\alpha = 0{\cdot}02$, $\bar{B} = 0{\cdot}2$. Asymptotic half-power law: —·—, $\alpha = \alpha_0 = 0{\cdot}04$. Experiments of Spangenberg *et al.* (1967): ○, station 130 distribution A; +, station 80 distribution A; $\alpha \simeq 0{\cdot}04$; $\alpha = 0{\cdot}02$.

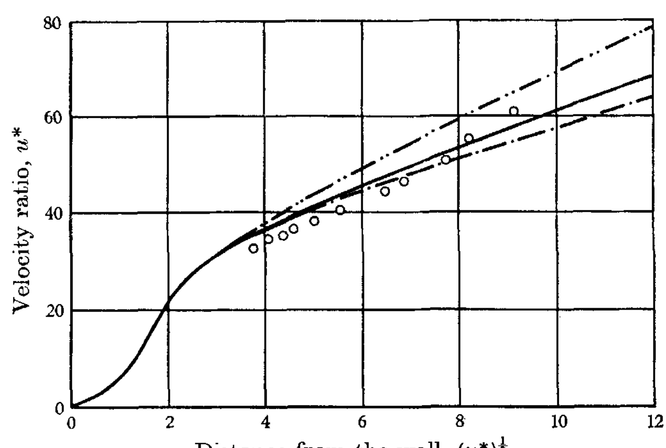

FIGURE 10. A comparison between a measured velocity profile of Patel and theory. Theory: $\alpha_0 = 0{\cdot}09$; —··—, $\alpha = 0{\cdot}09$; ———, $\alpha = 0{\cdot}045$, $\bar{B} = 0$; —·—, $\alpha = 0{\cdot}045$, $\bar{B} = 0{\cdot}2$; Experiments of Patel (1965); ○, $\alpha_0 = 0{\cdot}09$, α unknown.

In Patel's (1965) experiments, the surface stress was determined using a sublayer fence with a correction for pressure gradient included. The correction applied by Patel is, of course, open to question, but lacking anything nearly as accurate, the quoted shear stresses have been without modification. Unfortunately, no information is available from Patel's measurements to enable an estimate of the stress gradient to be made, even according to the procedure used in Perry's (1966) case. In the light of the measured stress gradients of both Spangenberg *et al.* (1967) and Newman (1951), it would be expected that the stress gradient in Patel's case would be nearer $0{\cdot}5\alpha_0$ than α_0. However, in figure 10 measured velocity profiles are compared with theory for both these values of stress gradient, and the results give an indication of the inaccuracy which results from being unable to determine the stress gradient rigorously.

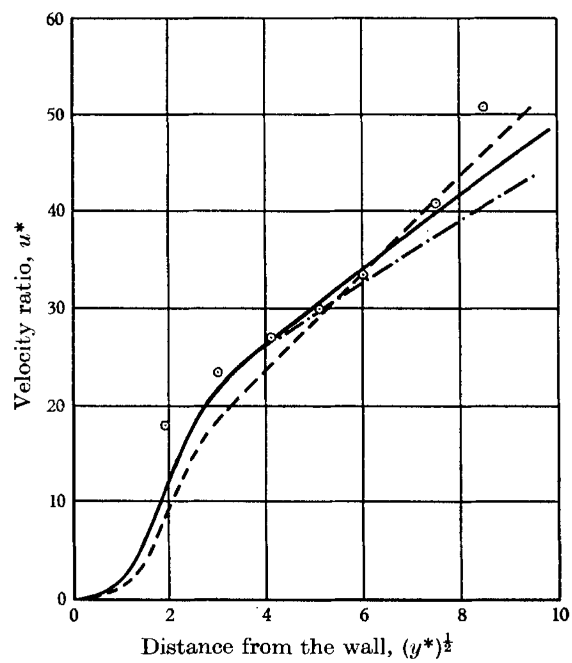

FIGURE 11. A comparison between a measured velocity profile of Stratford and theory. — — —, calculation performed by Mellor (1966*b*), $\alpha_0 = \alpha = 2{\cdot}0$; ———, present calculation, $\alpha = 0{\cdot}25$, $\alpha_0 = 0{\cdot}5$, $\bar{B} = 0{\cdot}0$; —·—·—, present calculation, $\alpha = 0{\cdot}25$, $\alpha_0 = 0{\cdot}5$, $\bar{B} = 0{\cdot}2$; ○, measurements of Stratford, reproduced from Mellor (1966*b*), α_0 and α unknown.

In Stratford's (1959*b*) experiments, the wall stress was nominally zero and once again the stress gradient unknown. As in Patel's case, the velocity profiles were calculated for a stress gradient equal to $0{\cdot}5\alpha_0$ and α_0. Stratford's nominally zero surface shear stress can only be regarded as a very approximate figure in the light of the considerable difficulties involved and the crude method Stratford was forced to use to determine when the point of zero skin friction has been obtained. Mellor (1966*b*) suggested a value of α_0 (which is tantamount to suggesting a skin friction value) for a particular velocity profile, and, in the present note, Mellor's suggestion is reproduced and compared with a profile calculated by arbitrarily using a slightly larger skin friction. The comparisons are presented in figure 11.

In general, the agreement between the measured and calculated velocity profiles, with or without the pressure-velocity diffusion contribution, is quite good. The inclusion of the pressure-velocity diffusion term had only a very small, favourable, effect on the comparisons. This small favourable effect must be weighted against the uncertainties in the stress gradient which could have had the same result. As a result, it seems advisable to postpone including the pressure-velocity diffusion effect on the velocity profile until more direct evidence as to its existence can be obtained.

Recently the work of Perry *et al.* (1966), and Perry (1966), has cast considerable doubt (unwarranted as it turns out) on, among other things, the additive constants used in Townsend's modified law of the wall. Indeed, it was thought by the present author at one time that a more precise evaluation of the additive constants, such as has been undertaken in the present note, would meet the objections raised by Perry and his co-workers, and this thought provided some stimulus for the present work. However, the results of the present investigation have shown, in agreement with Mellor (1966*b*), that in the low to moderate pressure gradient range under question by Perry *et al.*, the additive constants used by Townsend were, at least theoretically, quite reasonable. It follows that the answer to the criticism of Perry *et al.*, would seem to lie elsewhere, if at all. The criticism of Perry *et al.*, will now be examined in detail and shown to be unfounded.

At large values of the product of distance from the wall and the stress gradient, Townsend's (1961) modified law of the wall reduced to (34). According to Perry *et al.*, the available measurements do not agree with either the slope or the additive constant B of the half-power law given by (34). Perry *et al.* found empirically that the slope on a half-power law plot of a number of measured velocity profiles was given by $4{\cdot}17(a_0/\rho)^{\frac{1}{2}}$, rather than the values $2/\kappa_0(a_0/\rho)^{\frac{1}{2}}$ predicted by the modified law of the wall if the stress gradient, a, were equated to the pressure gradient, a_0. However, the present author found by surveying the available theoretical and experimental evidence (Newman 1951; Bradshaw 1966; Mellor 1966*a*), for mild pressure gradients in near equilibrium flows the stress gradient was generally about 0·7 of the pressure gradient. When this value is used in (34), excellent agreement is obtained between the predicted slope according to the modified law of the wall and the empirical results of Perry *et al.*

Turning to the question of B in (34), Perry *et al.* observed that the intercept of the $4{\cdot}17(a_0/\rho)^{\frac{1}{2}}$ line with $y = 0$ ordinate was considerably different from the appropriate value of B. Perry *et al.* claimed that these measured intercepts represented measured values of B, and the fact that they disagreed with theory cast considerable doubt on the theory. The measured intercepts are reproduced from Perry (1966) and compared to the theoretical values of B in figure 12. The key point here is that both Perry (1966) and Perry *et al.* (1966) measured the intercept of profiles which had not reached the asymptotic conditions required by the theory. This can readily be seen from, for example, figure 8, where two typical velocity profiles have been plotted. It can be seen that Perry *et al.* are in fact comparing their measurements with the asymptotic form of the modified law of the wall, the straight lines shown on figure 8, and evidently they ought to compare their measurements with the full versions of the modified law of the wall, since within the range of y presented there is still a considerable difference.

The theoretical profiles can only be expected to give reasonable answers as far as $y \sim 0{\cdot}2\Delta$, Δ being the boundary-layer thickness, in view of the mixing length model used in the analysis. The profiles in figure 8 never obtain the asymptotic state within this range of y less than 0·2 of the boundary-layer thickness, and this inability is a feature of most low pressure gradient boundary layers. Furthermore, both the theoretical and experimental profiles show a near linear behaviour on a square root abscissa in the pre-asymptotic state, thus making the empirical half-

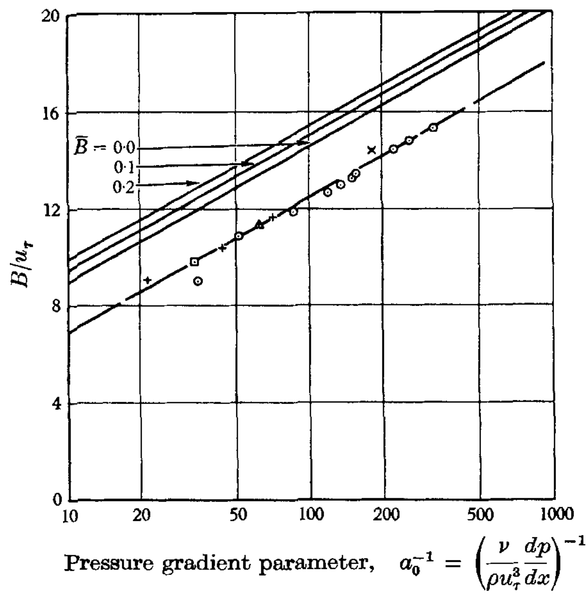

FIGURE 12. The slip velocity U_t/U_τ as a function of the streamwise pressure gradient. Measured slip velocities reproduced from Perry (1966). ⊙, Perry (1966); ×, Schubauer & Klebanoff (1951); ⊡, △, Perry, Bell & Jubert (1966); +, Johnston (1957); ———, Townsend theory (1961), $B^+ = 4{\cdot}9$, $A = 1{\cdot}0$; — —, Perry (1966).

power law of Perrry *et al.* a good fit to the measured profiles but, nevertheless, not really in contradiction to the modified law of the wall. It might also be noted that the foregoing is not a criticism of the velocity profile correlation of Perry *et al.*, since their correlation is empirical fact and stands as such. It is, however, a criticism of their objections to the modified law of the wall.

Another point which must be examined concerning the velocity profiles near the wall, is the disagreement observed between the empirical half-power law of Perry *et al.* and the profiles measured after a prolonged pressure rise, by Perry (1966). In this case, the velocity profiles initially were in good agreement with the empirical law, but, as the flow progressed, the agreement deteriorated, although the pressure gradient parameter, α_0, was small. As mentioned earlier, an estimate of the stress gradient was made using (23) with a value of b_w of 1/14. This gave a ratio of stress to pressure gradient of approximately 0·7, and the velocity profiles were then calculated according to the present note and are compared with Perry's measurements in figures 13 and 14. It can be seen that exactly the same trend of deteriorating agreement between theory and experiment as Perry found, using his semi-empirical velocity profile, is also obtained with the modified law of the wall. However, the measured velocity profiles do continue to vary as the square root of y, so that the observed departure from the predicted profiles could be explained by the actual ratio of stress to pressure gradient being much less than the assumed value of around 0·7. Since Perry did not measure the turbulent shear stress distribution this explanation can only remain speculative, but experience with similar types of boundary layers indicates that it is not an improbable suggestion. With this suggested explanation in

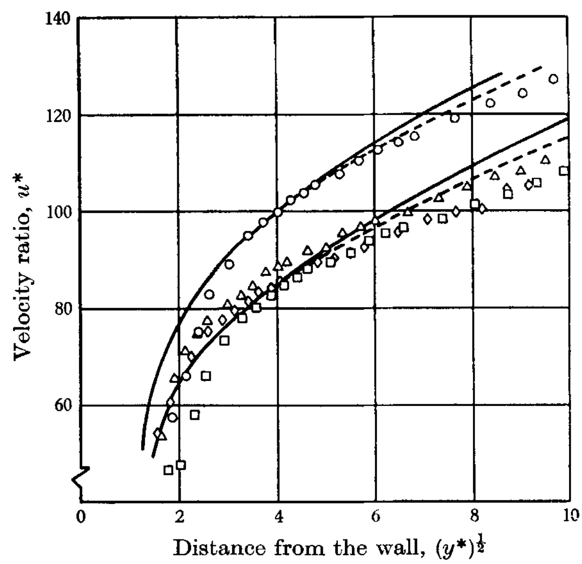

FIGURE 13. Further comparison between the measured velocity profiles of Perry and theory. Theory: $\alpha \simeq 0{\cdot}7\alpha_0$ and α_0 as measured; ———, $\bar{B} = 0{\cdot}0$; — — —, $\bar{B} = 0{\cdot}2$. Measurements of Perry (1966): ○, 5·5 ft. station, $\alpha_0 = 0{\cdot}00456$; □, 7 ft. station, $\alpha_0 \simeq 0{\cdot}007$; △, 8·5 ft. station, $\alpha_0 \simeq 0{\cdot}007$; ◇, 10 ft. station, $\alpha_0 \simeq 0{\cdot}007$.

mind, Newman's (1951) boundary-layer velocity profiles, which are very similar in many respects to Perry's, were examined. In this instance, the turbulent shear stress distribution normal to the wall was measured by Newman so that no problem arose in this respect. When the predicted and measured profiles were compared, good agreement was obtained throughout the length of the boundary-layer development, as may be seen in figure 15. As the separation point was approached in Newman's case, the ratio of stress to pressure gradient dropped lower than one-third and, if these sort of figures were applied in Perry's case, much better agreement between theory and experiment would result. Therefore, in the light of Newman's measurements, it would seem that quite accurate predictions of the velocity profiles near the wall can be made if the stress gradient is known, and that the unknown stress gradient is the real source of discrepancy between theory and experiment in the case of Perry's predictions.

Perry has attempted to explain the observed discrepancies mentioned in the previous paragraph by introducing a rate of change of pressure gradient contribution to the velocity profile. Obviously, the more local and higher-order conditions

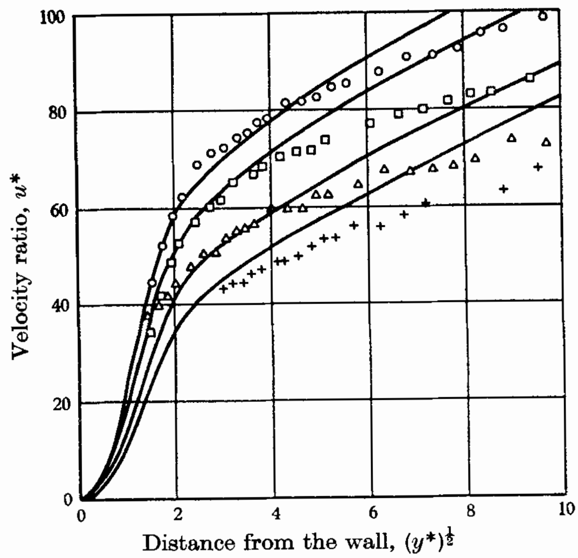

FIGURE 14. Further comparison between the measured velocity profiles of Perry and theory; continued. ———, theory $\alpha \simeq 0{\cdot}7\alpha_0$ and α_0 measured, $\overline{B} = 0$. ○, □, △, +, measurements of Perry (1966), stations $x = 11$, 12, 5, 14 and 15 ft., $\alpha_0 = 0{\cdot}0089$, 0·0114, 0·0196 and 0·025 respectively.

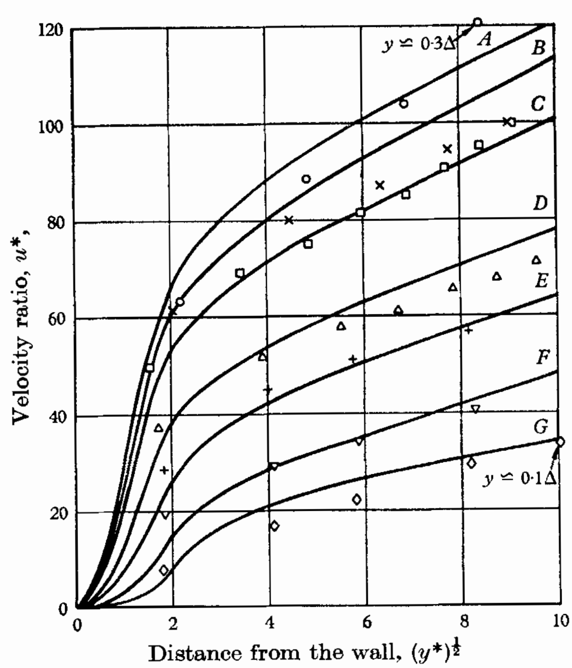

FIGURE 15. A comparison between the measured velocity profiles of Newman (1951) and theory. Measured: ○, station A, $C_f = 0{\cdot}00195$, $\alpha_0 = 0{\cdot}0064$, $\alpha = 0{\cdot}0035$; ×, station B, $C_f = 0{\cdot}00158$, $\alpha_0 = 0{\cdot}0083$, $\alpha = 0{\cdot}0052$; □, station C, $C_f = 0{\cdot}00117$, $\alpha_0 = 0{\cdot}0114$, $\alpha = 0{\cdot}0079$; △, station D, $C_f = 0{\cdot}00091$, $\alpha_0 = 0{\cdot}0246$, $\alpha = 0{\cdot}012$; +, station E, $C_f = 0{\cdot}0055$, $\alpha_0 = 0{\cdot}054$, $\alpha = 0{\cdot}0162$; ▽, station F, $C_f = 0{\cdot}00018$, $\alpha_0 = 0{\cdot}3$, $\alpha = 0{\cdot}1$; ◇, station G, $C_f = 0{\cdot}0$, $\alpha_0 = 10$, $\alpha = 9{\cdot}2$. Theory: ———, $\overline{B} = 0{\cdot}0$, α_0 and α as measured.

that are introduced to specify the velocity profile, the more likely it is that the profiles being correlated will have had a similar upstream history, and hence, have a similar stress gradient. As a result, an improved correlation ought to be obtained. This seems, to the present author, rather circuitous and it would almost certainly be much more rewarding to simply try and correlate the stress gradient. The previous remarks illustrate a basic objection the present author has to an entirely empirical approach, since it provides so little physical insight into the problem.

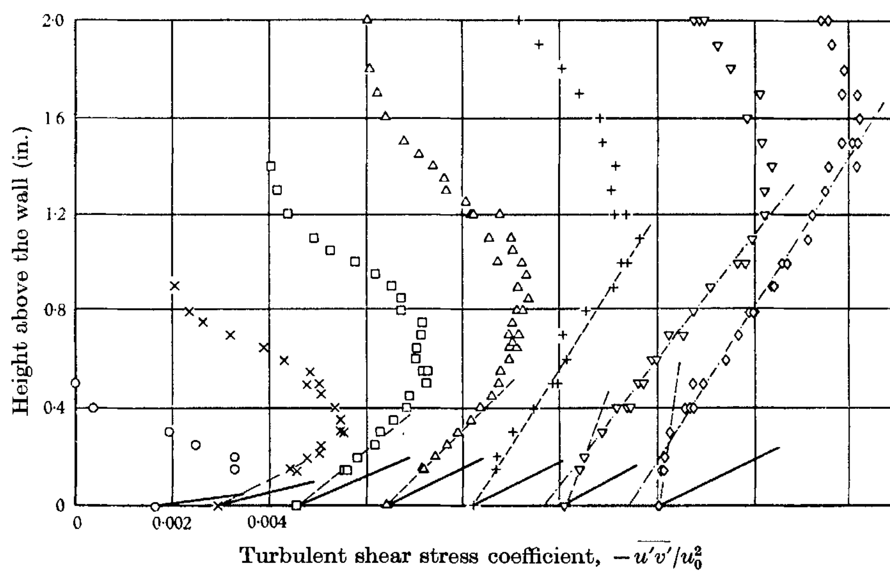

FIGURE 16. The measured shear stress profiles of Newman (1951). ———, $\partial\tau/\partial y = dp/dx$; ○, station *A*; ×, station *B*; □, station *C*; △, station *D*; +, station *E*; ▽, station *F*; ◇, station *G*.

The assumption of turbulent shear stress linearity near the wall is justified by the hot-wire measurements of turbulent shear stress made by Schubauer & Klebanoff (1951), Newman (1951), Bradshaw & Ferriss (1965), Bradshaw (1966), and Spangenberg *et al.* (1967). As a typical example, Newman's (1951), stress measurements previously used to construct the velocity profiles shown in figure 15, are presented in figure 16.

Lastly, no comparisons between predicted and measured velocity profiles in a favourable pressure gradient have been presented since in examining the strong favourable pressure gradient data see, for instance, Patel (1965), the problem of transition reversal is encountered and makes comparisons rather meaningless. In the case of the weak favourable pressure gradient data, no significant departures from the law of the wall are expected or observed (Patel 1965).

Conclusions

On the basis of the analysis described herein and comparisons with experimental evidence, the following conclusions have been reached.

The velocity profile across the viscous sublayer seems to be quite insensitive to possible changes in the stress gradient from its value at the wall. In other words, the so-called 'inertia effect' on the integrated velocity profile is small in the viscous sublayer. In contrast, however, in the fully developed turbulent flow region close to the wall, inertia effects are appreciable.

In view of previous conclusions, it would appear that a simple correction is possible to account for pressure gradient effects on any sublayer skin friction meter which depends for its calibration on the existence of a universal velocity profile family in the sublayer. Since inertia effects are not significant on the sublayer velocity profile, it would be expected that the additional calibration factor for such a skin friction meter would be simply the non-dimensional pressure gradient parameter, $\alpha_0 = (\nu/\rho u_\tau^3)(dp/dx)$.

Inclusion of the effect of pressure-velocity diffusion of turbulent energy in the modified law of the wall velocity profile was found to result in a small, generally favourable, improvement on the comparisons between predicted and measured velocity profiles outside the viscous sublayer. This small effect is not thought significant when compared to the uncertainties in the stress-pressure gradient relationship.

When Townsend's modified law of the wall is utilized with the additive constants of the present note, together with either measured or estimated stress gradients, good agreement between measured and predicted velocity profiles is obtained.

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